The physics of a row of toppling dominoes is discussed. The forces between the falling dominoes are analyzed, including the effect of friction. The propagation speed of the domino effect is calculated for the range of spatial separations for which the domino effect exists. The dependence of the speed as a function of the domino width, height, and interspacing is derived. © 2010 American Association of Physics Teachers.

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I. INTRODUCTION

The toppling of a row of dominoes provides a good illustration of the mechanics of solid bodies. The theory behind the domino effect is not as simple as the phenomenon might seem. For instance, the propagation velocity of the domino effect is not easy to calculate. In this paper we discuss the role of the collision law, the conservation laws, and the influence of friction. The crux is the form of the collision law on which the literature is surprisingly diverse.

Banks\(^1\) considered a row of toppling dominoes as a sequence of independent events: one domino undergoes a free fall until it hits the next one, which then falls independently of the others, and so on. He assumed that in the collision, the linear momentum along the supporting table is transmitted. We will show that this assumption is not consistent with experimental results. Efthimiou and Johnson\(^2\) also considered independent collisions between pairs of dominoes only in which kinetic energy and angular momentum are conserved so that their collisions are completely elastic. They found a propagation velocity that exhibits large deviations from the experimental values obtained by Larham.\(^3\) Shaw\(^4\) introduced a collision law that comes much closer to reality. He assumed that a domino, after having struck the next one, keeps pushing on it. His collision is therefore completely inelastic. In addition, he assumed conservation of angular momentum during the collision. Thus a train of dominoes that lean on each other and push the head of the train develops, which is what is observed and which is strongly supported by the high-speed recording of Stronge and Shu.\(^5\)

None of these authors treated the role of friction. In Ref. 4 the toppling of dominoes is treated as a collective effect of many dominoes simultaneously in motion. We take this viewpoint but show that for inelastic collisions, there is no room for conservation of angular momentum.

II. RECURSION RELATION

A theory of falling dominoes requires the use of idealized conditions. We consider a long row of identical and perfect dominoes of height \(h\), thickness \(d\), and interspacing \(s\). The lateral size of the dominoes is irrelevant. Their fall is due to the gravitational force with acceleration \(g\). As for all systems driven by gravitation, the mass of the dominoes drops out of the equations. Typical parameters of the problem are the aspect ratio \(d/h\), which is determined by the type of dominoes used, and the ratio \(s/h\), which can be easily varied in an experiment. Another characteristic of the dominoes is their mutual friction coefficient \(\mu\), which is a small number (~0.2).

The domino effect is a sequence of rotations interrupted by collisions. Both processes have dynamical laws, which are idealized as follows. The first assumption is that the dominoes only rotate and do not slip on the supporting table. While falling they keep contact with the supporting table. The no-slip condition can be realized through sufficient friction with the table. Stronge and Shu\(^5\) and Walker\(^6\) put them on sandpaper. The second assumption is the inelastic collision, after which the first and second dominoes stay in contact and fall together till they strike the third domino and so forth, making the domino effect a collective phenomenon.

Our main goal is to find the dependence of the propagation velocity on the parameters of the problem. Because \(g\) is the only quantity with time in its dimension, it may be anticipated that the speed will be proportional to \(\sqrt{gh}\) times a function of the dimensionless parameters \(s/h, d/h\), and \(\mu\). The speed initially depends on the initial push, but after a while a stationary pattern develops: A propagating wave with upright dominoes in front and toppled dominoes behind.

The typical situation is shown in Fig. 1. The state of domino \(i\) is given by the angle \(\theta_i\) with respect to the vertical. Before domino \(i\) is hit, \(\theta_i = 0\). The foremost moving domino is labeled by \(n\) with angle \(\theta_n\). At the moment of hitting the next one, \(\theta_n = \theta_i\) with \(\sin \theta_i = s/h\). As Fig. 1 shows, the angles of the dominoes behind the foremost one are dictated by the value of \(\theta_n\). The relation between the angle \(\theta_i\) and \(\theta_n\) follows from the expressions given in Table I. Equating the expression for BC to BD−\(d\) yields the recursion relation

\[
h \sin(\theta_i - \theta_{i+1}) = (s + d) \cos \theta_{i+1} - d. \tag{1}\]

We see that the angle \(\theta_i\) can be expressed in terms of \(\theta_{i+1}\). The recursion relation (1) defines all quantities as functions of \(\theta_n\). In most of the discussion this dependence is implicit in the notation, and we only make it explicit where it is useful (as, for instance, in Sec. V where the collision law is formulated).

Toward the back of the train of dominoes, the angles rapidly approach the stacking angle \(\theta_n\) given by \(\cos \theta_n = d/(s + d)\), making the right-hand side of Eq. (1) equal to zero. Thus, the falling dominoes have angles in the range of \(0 < \theta_i < \theta_n\). The foremost falling domino has an angle of \(0 < \theta_n < \theta_{i+1}\); the upper limit being the angle at which it loses its role as foremost moving.

Shaw\(^4\) observed that the angle \(\theta_n\) is the only independent dynamical variable; the other angles are dependent on \(\theta_n\) through Eq. (1). The peculiarity of the domino effect is that \(\theta_n\) is the independent variable only for the time that \(n\) is the foremost moving domino. Once \(n\) hits \(n+1\), it passes its role...
on to $\theta_{n+1}$ and becomes dependent on $\theta_{n+1}$. Thus, the role of independent variable is constantly shifting to the next angle.

Before performing the dynamical analysis, we calculate the “fuel” (energy source) of the domino effect. In the process each domino loses the difference in potential energy between the upright position and its value at the stacking angle $\theta_i$. We write this difference as $mghP/2$ to have a dimensionless measure $P$ ($m$ is the mass of a domino),

$$P(h, d, s) = 1 - \cos \theta_{n} - \frac{d}{h} \sin \theta_{n} = \frac{sh - d\sqrt{s^2 + 2sd}}{h(s + d)}. \tag{2}$$

There are two forms of dissipation consuming this fuel: The inelastic collision and the friction between the dominoes while sliding over each other. $P$ has to be positive to keep the train moving. It gives a lower limit for the spacing

$$s/h > 2(d/h)^3[1 - (d/h)^2]. \tag{3}$$

On the upper side there is the trivial restriction that the dominoes must be close enough, $s < h$, to touch each other in the fallen position. For mathematical convenience we impose a sharper restriction, that $h$ is large enough that a fallen domino can rest on the next one

$$h^2 > (s + d)^2 - d^2, \quad \text{or} \quad s/h < \sqrt{1 + (d/h)^2} - d/h. \tag{4}$$

For larger separations the recursion relation is not satisfied for the dominoes at the back of the train. We will argue that for such large separations, the no-slip condition makes little sense.

It is interesting to relate condition (4) to the approach of the angles to $\theta_{n}$. Let $\theta_{n+1}$ be a small angle $\delta_{n+1}$ away from the stacking angle $\theta_{n}$. Then, according to Eq. (1), $\delta_{n}$ equals

$$\delta_{n} = \left(1 - \frac{s + d}{h} \sin \theta_{n}\right)\delta_{n+1}. \tag{5}$$

The factor between brackets never exceeds 1, so the tilt angles approach the stacking angle exponentially fast. Thus, although the row of fallen dominoes becomes arbitrarily long, the length of the set of falling dominoes is effectively finite and determined by the logarithm of the factor in Eq. (5). The condition Eq. (4) prevents the factor from becoming negative. If Eq. (4) is violated, the recursion relation as given in Eq. (1) does not hold anymore. So the approach to the stacking angle is monotonic and not alternating.

The domino effect is an alternation of rotations and collisions. In the rotation phase we have to calculate the final angular velocity of domino $n$ from its initial value. The collision connects the final angular velocity of $n$ with the initial value of domino $n+1$. We first discuss the equations for frictionless rotation and analyze the forces between sliding dominoes as a warm-up exercise for the collision analysis. We then formulate the precise collision law, inspect the conservation laws, and the different collisions laws assumed in the literature. The next step is the introduction of friction and the calculation of its effect on the propagation velocity. Then we present our results for the asymptotic velocity for various values of the friction and compare them with experiment. We also give an explicit expression for the main dependence of the velocity on the parameters of the system. The paper closes with a discussion of the results and the assumptions that we have made.

### III. ROTATION EQUATIONS

In this section the equation for the rotating angle $\theta_i$ is given in the time interval that $n$ is the foremost moving domino. Thus, the other variables $\theta_i$ have to be eliminated. As mentioned, we take the index $n$ to designate the foremost falling domino and the index $i$ for the domino number counting from the beginning. Thus, $i$ is in the interval $1 \leq i \leq n$.

We also introduce the index $j$ for the domino number relative to the foremost falling one. Thus, $j$ is in the interval $0 \leq j < n$. A bar over the symbols refers to quantities summed over $i$ or $j$.

Sums over $i$ or $j$ run over their respective intervals unless otherwise stated. It is useful to employ both types of numbering of the dominoes. For instance the recursion, implied by Eq. (1) is more conveniently represented by therenumbering

$$\theta_{n}(\theta_{n}) = \psi_{n-n}(\theta_{n}) = \psi_{n}(\theta_{n}). \tag{6}$$

The functions $\psi_{j}(\theta)$ and their derivatives play an important role in the elimination process. A number of their properties are listed in the Appendix. For each value of $\theta$, $\psi_{j}(\theta) = \theta$ (by definition) and converges rapidly to $\psi_{j}(\theta) = \theta_{j}$ for large $j$. Here we assume the $\psi_{j}$ and their derivatives as known functions.

A consequence of Eq. (6) is the expression for the angular velocities $\omega_{n} = d\theta_{n}/dt$ in terms of $\omega_{n}$. We find

$$\omega_{n} = \frac{d\theta_{n}}{dt, \omega_{n}} = \psi_{j}^{n}, \omega_{n}, \tag{7}$$

with $\psi_{j}^{n} = d\psi_{j}(\theta)/d\theta$.

Without friction, the motion between two collisions is governed by conservation of energy, which consists of a po-
potential and a kinetic part. The potential part \( V_n \) is determined from the height of the center of mass and is given by
\[
V_n = (mgh/2)H_n = (mgh/2)\sum_i [\cos \theta_i + (d/h)\sin \theta_i].
\]
(8)

The factor \( mgh/2 \) is an irrelevant energy scale. The kinetic part \( K_n \) is given by the rotational energy,
\[
K_n = I_n \omega_n^2,
\]
(9)
where \( I = m(h^2 + d^2)/3 \) is the angular moment of inertia with respect to the rotation axis. We write the total (dimensionless) energy as
\[
E_n = H_n + \omega_n^2 I_n,
\]
where the (dimensionless) effective moment of inertia \( \tilde{I}_n \) is defined as
\[
\tilde{I}_n = \sum_j \psi_j^2
\]
(11)
and the time scale \( \tau \) for the domino effect is
\[
\tau = \sqrt{I/mgh}.
\]
(12)

Because \( E_n \) is constant in the rotational interval, Eq. (10) gives \( \omega_n \) as function of \( \theta_n \),
\[
\omega_n(\theta) = \frac{1}{\tau} \left( \frac{H_n(0) + \omega_n(0)^2 \tilde{I}_n(\theta) - H_n(\theta)}{\tilde{I}_n(\theta)} \right)^{1/2},
\]
(13)
where we have expressed \( E_n \) in terms of the initial angular velocity \( \omega_n(0) \) and height \( H_n(0) \). The temporal behavior of \( \theta_n \) is found from the relation
\[
\frac{d\theta_n(t)}{dt} = \omega_n(\theta_n(t)),
\]
(14)
with \( \omega_n(\theta) \) determined by Eq. (13). Inversely we find the time \( t \) as function of \( \theta \). The initial value of \( \theta_n \) is 0, and its final value is \( \theta \). The time interval during which \( n \) is the head of the train follows by integration,
\[
t_n = \int_0^\theta \frac{d\theta'}{\omega_n(\theta')}.
\]
(15)

In this time interval the position of the foremost moving domino has advanced the distance \( s + d \). The ratio \( v_n=(s + d)/t_n \) gives the velocity for the time that \( n \) is the head of the train. Thus, the problem is reduced to finding \( \omega_n(0); \omega_i(0) \) is determined by the initial push. The relation between \( \omega_i(0) \) and \( \omega_i(\theta_i) \) is given by the collision law and so on for the subsequent stages.

IV. FORCES BETWEEN FRICTIONLESS SLIDING DOMINOS

Before we introduce friction we take a closer look at the forces between the falling dominos. Without friction the force that domino \( i \) exerts on domino \( i + 1 \) is perpendicular to the surface of domino \( i + 1 \) with a magnitude \( f_i \) (see Fig. 1). Apart from these mutual forces, domino \( i \) feels the gravitational torque \( T_i \).
\[
T_i = mg(h \sin \theta_i - d \cos \theta_i)/2.
\]
(16)

Consider the head of the train. It feels a gravitational torque \( T_n \) and a torque from domino \( n - 1 \) equal to the force \( f_{n-1} \) times the moment arm \( a_{n-1} \) with respect to the rotation point of domino \( n \). The angular acceleration of domino \( n \) thus becomes
\[
I \frac{d\omega_n}{dt} = T_n + f_{n-1}a_{n-1}.
\]
(17)

Domino \( n - 1 \) feels, besides the gravitational torque \( T_{n-1} \), a torque from \( n \), which slows it down, and a torque from \( n - 2 \), which speeds it up. The equation for domino \( i \) has the form
\[
I \frac{d\omega_i}{dt} = T_i + f_{i-1}a_{i-1} - f_ib_i.
\]
(18)

The torque due to the force \( f_{i-1} \) of domino \( i - 1 \) on domino \( i \) has a moment arm \( a_{i-1} \) with respect to the rotation point of domino \( i \). From Newton’s third law, the force of domino \( i + 1 \) on domino \( i \) is the negative of the force that domino \( i \) exerts on domino \( i + 1 \). The torque has the associated moment arm \( b_i \). The moment arms are given in Table I,
\[
b_i = h \cos(\theta_i - \theta_{i+1}), \quad a_i = b_i - (s + d) \sin \theta_{i+1}.
\]
(19)

Note that Eq. (17) is a special case of Eq. (18) with \( f_n=0 \). For \( i=1 \), Eq. (18) holds with \( f_0=0 \). There are only \( n-1 \) mutual forces ranging from \( f_{n-1} \) to \( f_1 \). Remember that the \( a_i \) and \( b_i \) are functions of \( \theta_i \) through the recursion relation (1). We see by definition that \( a_i < b_i \). Thus, domino \( i \) gains less from domino \( i-1 \) than domino \( i-1 \) loses to domino \( i \).

Although the moment arms are explicitly given, the forces \( f_i \) are unknown. We can eliminate the forces from the equations because each \( f_i \) occurs only in two successive equations. Thus if we multiply Eq. (17) by \( r_0=1 \) and the general equation by \( r_{n-1} \), the total sum vanishes,
\[
\sum_{i=1}^n r_{n-i} \left( \frac{d\omega_i}{dt} - T_i \right) = 0,
\]
(20)

provided that the \( r_i \)’s are chosen such that
\[
r_{n-1}b_i = r_{n-1-i}a_i.
\]
(21)

Equation (20) can be converted into an equation for the angular velocity \( \omega_n \). We differentiate Eq. (7) with respect to the time,
\[
\frac{d\omega_n}{dt} = \psi_{n-1} \omega_{n-1}^2 + \psi_{n-1} \frac{d\omega_n}{dt},
\]
(22)

and rewrite the time derivative of \( \omega_n \) as
\[
\frac{d\omega_n}{dt} = \frac{d\omega_n}{d\theta_n} \frac{d\theta_n}{dt} = \frac{d\omega_n}{d\theta_n} \omega_n = \frac{1}{2} \frac{d\omega_n^2}{d\theta_n}.
\]
(23)

Equation (20) becomes
\[
\frac{1}{2} A_n(\theta) \frac{d\omega_n^2}{d\theta} + B_n(\theta) \omega_n^2 = C_n(\theta),
\]
(24)

with the definitions
\[
A_n(\theta) = \sum_j r_j \psi_j', \quad B_n(\theta) = \sum_j r_j \psi_j^2', \quad C_n(\theta) = \sum_j r_j T_{n-j}.
\]
(25)
Equation (24) is a linear first-order differential equation for $\omega_n$, which can be solved in terms of integrals over the coefficients $A$, $B$, and $C$. Because these integrals have to be evaluated numerically, it hardly pays to use such an explicit solution rather than straightforwardly integrate the equation numerically. With $\omega_n$ known as a function of $\theta$, we can perform the integration in Eq. (15) for the duration of the interval $n$.

V. THE COLLISION EQUATIONS

The stages of rotational motion are connected by collisions. In this section we calculate the initial value $\omega_0(0)$ of stage $n$ from the final value $\omega_{n-1}(\theta_i)$ of the previous stage. The idea is that during the collision, forces are exerted during a very short time interval such that the angles do not change during the collision. Instead the angular velocities make a jump. When domino $n-1$ hits $n$, its own angular velocity is suddenly reduced, and that of $n$ jumps to the nonzero value $\omega_n(0)$. The jumps in the angular velocity decrease in magnitude as the collisions propagate down the train in such a way as to keep the dominoes in contact. Therefore the impulses have to decrease to realize these jumps. For domino $i$ the jump in angular velocity is

$$\Delta \omega_i = \psi_n(0)\omega_n(0) - \psi_{n-1}(\theta_i)\omega_{n-1}(\theta_i).$$

(26)

The first term is the angular velocity as calculated from $n$ just after the collision, and in the second it is calculated from $n-1$ just before the collision. For $i=n$ the second term is absent and $\psi_n(0)=1$. We denote the impulses by $F$: Domino $i$ receives $-F_i$ from $i+1$ and $F_{i-1}$ from $i-1$. Again using Newton’s third law we have

$$\Delta \omega_i = F_{i-1}a_{i-1} - F_ib_i,$$

(27)

Equation (27) holds also for $i=n$ with $F_n=0$. For $i=1$ we must set $F_0=0$. The functions $a_i$ and $b_i$ are defined in Eq. (19). The impulses $F_i$ can be eliminated in the same way as before by multiplying the $i$th equation by $r_{n-i}(0)$ and summing them. For the coefficient of $\omega_n(0)$ we obtain

$$\sum_{i} r_{n-i}(0)\psi'_{n-i}(0) = \sum_{j} r_j(0)\psi'_j(0) = A_n(0),$$

(28)

with $A_n$ defined in Eq. (25). For the coefficient of $\omega_{n-1}(\theta)$, we find using Eq. (A3)

$$\sum_{i=1}^{n-1} r_{n-i}(0)\psi'_{n-i-1}(\theta_i) = \sum_{i=1}^{n-1} r_{n-i}(0)\psi'_{n-i}(0)$$

$$+ \sum_{j=1}^{n-1} r_j(0)\psi'_j(0) = A_n(0) = 1.$$ 

(29)

Thus, the final collision law reads

$$A_n(0)\omega_n(0) = [A_n(0) - 1]\omega_{n-1}(\theta_i),$$

(30)

which expresses the initial value of $\omega_n(0)$ in terms of the value $\omega_{n-1}(\theta_i)$ of the preceding domino just before the collision.

VI. CONSERVATION LAWS

The frictionless forces between the dominoes, as discussed in Sec. IV, conserve energy during the rotation. Note that the recursion relation Eq. (21) for $r_j$ is the same as that for the $\psi'_j$ in Eq. (A5). They begin as $r_0=1$ and $\psi'_0=1$, and therefore we may identify $r_j = \psi'_j$. This identification implies that $A_n$ defined in Eq. (25) and $\bar{I}_n$ defined in Eq. (11) are equal,

$$A_n(\theta) = \bar{I}_n(\theta).$$

(31)

Thus multiplying Eq. (20) by $\omega_n$ and using that $r_{n-1}\omega_n = \psi'_{n-1}\omega_n = \omega_n$ (see Eq. (7)) give

$$\frac{d}{dt}\left[\frac{1}{2}\sum_{i} \omega_i^2\right] = \sum_{i} \omega_i T_i.$$ 

(32)

From Eqs. (8) and (16) we see that

$$\frac{d}{dt}(mgh/2) V_n = -\sum_{i} \omega_i T_i,$$

(33)

which turns Eq. (32) into the standard form of conservation of energy.

We use Eq. (31) to write the collision law (30) as

$$\bar{I}_n(0)\omega_n(0) = (\bar{I}_n(0) - 1)\omega_{n-1}(\theta_i) = \bar{L}_{n-1}(\theta_i)\omega_{n-1}(\theta_i).$$

(34)

We have added the last equality (using Eq. (A3)) because it leads to an equation of the same form as the conservation of angular momentum with the effective angular moment of inertia $\bar{I}_n$. This moment of inertia is linked to the energy and not to the angular momentum. The angular momentum $\bar{L}_n$ of a train of moving dominoes up $n$ is given by

$$\bar{L}_n(\theta_i) = I\sum_{i} \psi'_{n-i}(\theta_i)\omega_i(\theta_i).$$

(35)

Shaw assumed that angular momentum is conserved. Thus instead of Eq. (30), he used the relation

$$\bar{L}_n(0) = \bar{L}_{n-1}(\theta_i).$$

(36)

Because $\bar{L}_n(\theta)/\bar{I}_n(\theta)\omega_n(\theta)$, the collision law (36) leads to a larger propagation velocity than our collision law (30). The difference between Eqs. (30) and (36) follows from the difference between the moment arms $a_i$ and $b_i$, which in turn is due to the fact that the two colliding dominoes do not rotate around the same point. If $a_i$ were equal to $b_i$, then all $r_i=1$ and $\psi'_j$ would have entered in Eq. (29) to the first power as in $\bar{L}_n$.

In contrast, Banks postulated conservation of linear momentum along the supporting table, which amounts to the collision relation

$$\omega_n(0) = \cos \theta_i \omega_{n-1}(\theta_i).$$

(37)

The factor $\cos \theta_i$ accounts for the horizontal component of the linear momentum. This collision law gives a slower velocity because the (omitted) collective effects speed up the domino effect.

Although the rotation is non-dissipative if friction is neglected, kinetic energy is dissipated in the collisions. Because we take the collisions as instantaneous, the potential energy is the same before and after the collision. Before the collision the total kinetic energy equals

$$\bar{K}_{n-1}(\theta_i) = \frac{1}{2} \sum_{i=1}^{n-1} \omega_i^2 = \frac{1}{2} \sum_{i=1}^{n-1} \psi'_{n-1-i}(\theta_i)\omega_i^2(\theta_i),$$

(38)

and after the collision it equals
Table II. The asymptotic propagation velocity (with $d/h=0.179$) for the collision law of Shaw (Ref. 4) and for various degrees of friction with the collision law in Eq. (30). The last column gives the results according to the collision law of Banks (Ref. 1).

<table>
<thead>
<tr>
<th>$s/h$</th>
<th>Shaw*</th>
<th>Frictionless</th>
<th>$\mu=0.1$</th>
<th>$\mu=0.2$</th>
<th>$\mu=0.3$</th>
<th>Banks$^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3.64568</td>
<td>2.23469</td>
<td>1.82095</td>
<td>1.51695</td>
<td>1.28221</td>
<td>⋯</td>
</tr>
<tr>
<td>0.2</td>
<td>3.06742</td>
<td>1.95534</td>
<td>1.66019</td>
<td>1.43423</td>
<td>1.25452</td>
<td>⋯</td>
</tr>
<tr>
<td>0.3</td>
<td>2.74686</td>
<td>1.82515</td>
<td>1.56987</td>
<td>1.37279</td>
<td>1.21498</td>
<td>⋯</td>
</tr>
<tr>
<td>0.4</td>
<td>2.50849</td>
<td>1.74231</td>
<td>1.50459</td>
<td>1.32193</td>
<td>1.17605</td>
<td>0.70506</td>
</tr>
<tr>
<td>0.5</td>
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<td>1.67865</td>
<td>1.44771</td>
<td>1.27272</td>
<td>1.13420</td>
<td>0.91875</td>
</tr>
<tr>
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</tr>
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<td>1.33347</td>
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</tr>
<tr>
<td>0.8</td>
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<td>0.94711</td>
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</tr>
<tr>
<td>0.9</td>
<td>1.46454</td>
<td>1.47984</td>
<td>1.15267</td>
<td>0.95822</td>
<td>0.82681</td>
<td>0.76941</td>
</tr>
</tbody>
</table>

*Reference 4.
$^b$Reference 1.

\[
\bar{K}_n(0) = \frac{I}{2} \sum_{i=1}^{n} \alpha_i^2 = \frac{I}{2} \sum_{i=1}^{n} \mathcal{Y}_{i-1}^2(0) \omega_i^2(0).
\]  

(39)

We can rewrite these sums as in Eq. (29) and then find using Eqs. (30) and (31) that the ratio

\[
\frac{\bar{K}_n(0)}{\bar{K}_{n-1}(\theta_i)} = \frac{I_{n}(0)}{I_{n}(0)} - 1
\]

(40)

is always smaller than 1. For small separations many dominoes participate effectively in the train of falling dominoes, which makes $I_n(0)$ large and the transmission of kinetic energy effective. For large separations $I_n(0)$ reaches its limiting value equal to 2, and the kinetic energy is reduced by $1/2$ in each collision.

VII. FRICTION

After all this groundwork it is simple to introduce friction. Let us start with the equation of motion (18). Friction adds a force parallel to the surface of $i+1$. For the strength of the friction force, we assume the law of Amonton–Coulomb,

\[
f_{\text{friction}} = \mu f,
\]

(41)

where $f$ is the corresponding perpendicular force. The inclusion of friction means that the coefficients $a_i$ and $b_i$ pick up a frictional component. The associated torques follow from the geometry in Fig. 1. The values of the $a_i$ and $b_i$ change to

\[
a_i = h \cos(\theta_i - \theta_{i+1}) - (s + d) \sin \theta_{i+1} - \mu d,
\]

(42)

\[
b_i = h \cos(\theta_i - \theta_{i+1}) + \mu h \sin(\theta_i - \theta_{i+1}).
\]

With these new moment arms, the calculation is essentially reduced to the frictionless case discussed in Sec. IV. We may eliminate the forces as before, which again leads to Eq. (20), but with coefficients $\bar{r}_j$, which are expressed in terms of $\bar{a}_j$ and $\bar{b}_j$ as in Eq. (21), yielding a differential equation for $\bar{\omega}_j^2$ as in Eq. (24), with coefficients $\bar{A}$, $\bar{B}$, and $\bar{C}$ defined with $\bar{r}_j$ as in Eq. (25).

The inclusion of friction in the collision law is simpler because Eq. (29) remains valid, with the definitions (42) for $\bar{a}_i$, $\bar{b}_i$, and consequently $\bar{r}_j$.

VIII. CALCULATIONS AND EXPERIMENTS

To calculate the domino effect, we have to solve three equations repeatedly. We start with the initial rotation velocity $\omega_i(0)$ and (1) solve Eq. (24) for $\omega_i(\theta_i)$ up to $\omega_i(\theta_{i+1})$, (2) simultaneously integrate Eq. (15) for the duration $t_i$, and (3) use $\omega_i(\theta_i)$ and Eq. (30) to find $\omega_{i+1}(0)$. We repeat this cycle from $i=1$ to a value where $t_i$ becomes constant, yielding the asymptotic velocity $v_m$ of the domino effect. The approach is fast due to the exponentially fast approach of $\theta_i$ to the stacking angle $\theta_s$, as Eq. (5) shows. Therefore the region of falling dominoes is finite, separating the fallen dominoes from the untouched ones.

In Table II we summarize the results for $v_m/\sqrt{gh}$. The thickness to height ratio is set to $d/h=0.179$, which is standard for commercial dominoes and corresponds to values used in experiments. The first column gives the separation $s/h$, the second gives the velocity using Shaw’s collision law (35), and the third gives the results for our Eq. (30). In the subsequent columns the influence of friction is shown. Note that the reduction of the speed due to the change of the collision law is larger than that of modest friction. The last column gives the result for the collision law (36) of Banks. These velocities are too small, given the fact that friction is absent. Moreover, there is a high threshold in separation for the domino effect. In the approach of Banks it is given by the difference in potential energy of an upright domino and a domino that hits the next one at the angle $\theta_s$,

\[
P_R = 1 - \cos(\theta_s) - (d/h) \sin(\theta_s) = 1 - \sqrt{1 - \left(\frac{s}{h}\right)^2} - ds/h^2,
\]

(43)

where the factor $mgh/2$ is taken out as in Eq. (2). It leads to the threshold $s > 2d/(1+(d/h)^2)$, which is much higher than our condition (3). The reason is that after the collision, the domino following the head is out of the game, while in the collective treatment it keeps pushing. Dominoes with a finite thickness initially increase their potential energy by rotating. Banks ignored this potential barrier and treated the dominoes as if $d=0$. In this limit a more reasonable curve is obtained, which Larham used to compare with his experiments.

In Fig. 2 we plot our curves for $\mu=0.1$ and $\mu=0.2$ together with experimental values. The value $\mu=0.2$ follows from an estimate of the angle of the supporting table at which dominoes start to slide over each other; it agrees with
the value given by Stronge and Shou. The value μ = 0.1 comes close to the experimental data of Larham and those of Stronge and Shu. Stronge and Shu used thinner dominoes (d/h = 0.12), but the differences between the theoretical curves for d/h = 0.179 and d/h = 0.12 are too small to plot. Those of McLachlan et al. are systematically lower and suggest a somewhat larger friction. These authors also suggested that the velocity diverges for small separations, which does not occur for finite d, because the lower limit (3) will be approached where the domino effect runs out of fuel. We found that the theoretical curve bends over sharply near the threshold (which is very low for d/h = 0.179). This bend-over is more theoretical than practical because close to the threshold, the first domino needs a large initial rotation to start the domino effect to overcome the initial rise in potential energy. Note that the domino effect still exists below the threshold of Banks, which is a clear demonstration of the collective character of the domino effect.

IX. SIMPLIFICATIONS

With friction there is no way of calculating the propagation velocity other than numerical integration of the dynamic equations. Without friction simplifications are possible, which provide further insight in the domino effect. In the frictionless case we can use Eq. (13) instead of integrating Eq. (24) to find the θ dependence of ωn. In Eq. (34) we saw how the collision law simplifies without friction. In addition, we know that the stationary state does not depend on the index n of the foremost moving domino. Then conservation of energy during the rotation from θ = 0 to θ = θc can be used in the form

\[ \tau^2 [\ddot{\theta} \omega^2(\theta_c) - \ddot{\theta} \omega^2(0)] = \ddot{H}_n(0) - \ddot{H}_n(\theta_c) = P(h,d,s). \] (44)

The left-hand side is the increase in kinetic energy, and the absence of the index n indicates that the stationary state has been reached. The right-hand side is the decrease in potential energy. We cannot drop the index n on \( \ddot{H}_n \), because \( \ddot{H}_n \) increases with n, but the difference becomes independent of n, as the second equality in Eq. (44) indicates. This equality follows by comparing both terms with \( \ddot{H}_{n+1}(0) \). \( \ddot{H}_n(0) \) is smaller than \( \ddot{H}_{n+1}(0) \) by one domino at the stacking angle, and \( \ddot{H}_n(\theta_c) \) is smaller than \( \ddot{H}_{n+1}(0) \) by one domino in the upright position. \( P(h,d,s) \) is the fuel of the domino effect defined in Eq. (2). Not surprisingly the fuel is consumed in the collision, as the drop of the kinetic energy, following from Eqs. (38) and (39), shows.

We solve Eqs. (34) and (44) for \( \omega(0) \) and \( \omega(\theta_c) \) and find

\[ \omega^2(0) = \frac{P(h,d,s)}{\tau^2} - \ddot{I}_0, \quad \omega^2(\theta_c) = \frac{P(h,d,s)}{\tau^2} - \ddot{I}_0 - 1. \] (45)

Thus for a fully developed domino effect, the initial and final angular velocities are explicitly known, and only the integration of Eq. (14) has to be done. Equation (45) shows that \( \tau \) is the dominant influence on the magnitude of \( \omega \). For small separations many dominoes participate in the train, and \( \ddot{I}_0 \) is large and drops out. The minimum value for \( \ddot{I}_0 \) is 2 and is reached for large separations. The fully developed domino effect becomes more transparent by introducing the average

\[ \langle \omega \rangle^{-1} = \theta_c^{-1} \int_0^{\theta_c} \frac{d\theta}{\omega(\theta)} \] (46)

with \( \omega(\theta) \) as the solution of Eq. (13). The value of the average \( \langle \omega \rangle \) is close to \( \sqrt{\tau P/\tau} \) because the integrand varies from a value slightly larger than \( \tau \sqrt{P} \) to a value slightly less than \( \tau \sqrt{P} \). Using this average in Eq. (15) gives the asymptotic velocity

\[ v_m = \sqrt{ghQ(h,d,s)} \frac{\langle \omega \rangle \tau}{\sqrt{P(h,d,s)}}, \] (47)

where the factor \( Q \) is given by

\[ Q(h,d,s) = \left( \frac{3}{1 + d^2/h^2} \right)^{1/2} \sqrt{\frac{(s + d) P(h,d,s)}{h \arcsin(s/h)}}. \] (48)

\( Q \) is the main factor that determines the dependence of the velocity on the parameters of the problem. The fraction in Eq. (47) is a refinement that requires a detailed calculation. We find that this fraction is almost independent of the aspect ratio \( d/h \) and remains close to 1 for the major part of the range of practical separations. Only near the already unworkable separation \( s/h = 0.9 \) does the value increase by about 10%.

X. DISCUSSION

We have derived a set of equations for the domino effect of a row of equally spaced dominoes under the assumptions that the dominoes only rotate and that they keep leaning onto each other after a collision with the next one. The treatment is close to that of Shaw, who introduced the constraint (1), which synchronizes the motion of the train of toppling dominoes. By analyzing the mutual forces between the dominoes, we correct his collision law and also account for the effect of friction between the dominoes. The correction of the collision law is more important than the influence of friction, given the small friction coefficient between dominoes. Equations (47) and (48) display explicitly the main dependence of...
the velocity on the parameters of the problem. The maximum speed is close to the smallest separation (see Eq. (3)) for which the domino effect exists. Friction always slows down the domino effect.

The domino effect produces an effectively non-dispersive, localized, propagating wave. Therefore it shares some features with a soliton wave: The constant propagation velocity and the invariant wave profile. The main difference is that a soliton is non-dissipative, while dissipation plays an important role in the domino effect. Thus, while differing mechanistically from other solitons, it is an interesting example of similar behavior.

We have imposed for simplicity the bound \( \beta \). It is not very interesting to sort out what happens if Eq. (4) is violated, as the no-slip condition becomes highly questionable near this bound. For such wide separations the force on the struck domino has hardly a torque to rotate it. As a practical limitation we propose that the height of impact remains above the center of mass of the struck domino. For an impact below the center of mass, the domino would slip over the table and topple over in the wrong direction. Insisting under these circumstances on the no-slip condition becomes unrealistic. This criterion yields the bound for the separation \( s/h < \sqrt{3}/2 = 0.87 \). Larham\(^3\) found the domino effect to disappear near this value.

The assumption of fully elastic collisions would yield an ever increasing velocity because there is no dissipation mechanism. In that case friction cannot play a role because the dominoes only touch during the collision. Ethimiou and Johnson\(^3\) and Banks\(^4\) found a finite propagation speed with elastic collisions because they ignored the energy release of the dominoes behind the foremost moving one.

In the less extreme case of partially elastic collisions, the dominoes also rotate without permanent contact, but friction can play a role during the collision. Stronge and Shu\(^6\) experimented with dominoes with restitution coefficients of the order of 0.6. Thus, instead of leaning onto each other, there is a sequence of dying-out collisions between the dominoes like a ball bouncing on the floor. As their recordings show, it is difficult to distinguish a sequence of rapidly dying out collisions from the inelastic collisions obeying Eq. (1). Note that friction adds to the validity of Eq. (1), as can be checked by pushing objects over tables with different friction coefficients. The analysis of such frequently colliding dominoes is difficult to distinguish a sequence of rapidly dying out collisions from the inelastic collisions obeying Eq. (1). Note that friction adds to the validity of Eq. (1), as can be checked by pushing objects over tables with different friction coefficients. The functions are strongly interrelated. Not only can we calculate \( \Psi_j(\theta) \) but \( \Psi_j(\theta) \) follows also from an arbitrary intermediate \( \Psi_k \) by \( \Psi_k \).

\[
\Psi_{n+1}(\theta) = \Psi_{k+1}(\theta), \quad \text{e.g.} \quad \Psi_j(\theta) = \Psi_{j-1}(\theta) + \Psi_{j+1}(\theta).
\]  

At the moment that domino \( n+1 \) sets \( n \) into motion, it has the angle \( \theta \). Thus \( \Psi_j(0) = \theta \), which implies

\[
\Psi_j(0) = \Psi_{j-1}(\theta_j(0)) = \Psi_{j-1}(\theta_j),
\]  
a property that will be used several times.

Determination of Eq. (11) with respect to \( \theta \) yields

\[
\frac{d\Psi_j}{d\theta} = \frac{1}{h} \left( 1 - \frac{(s + d)\sin \Psi_{j-1}(\theta)}{\cos \Psi_j(\theta) - \Psi_{j-1}(\theta)} \right).
\]  
The relevance becomes evident when we express the right-hand side in terms of the moment arms defined in Eq. (19),

\[
\Psi_j = \Psi_{j-1}(a_{n-j}/b_{n-j}),
\]  
which implies that the frictionless forces conserve energy as shown in Sec. VI. Another differentiation of Eq. (A4) gives the recursion relation between the second derivatives \( \Phi_j \). It can be cast into the form

\[
\Phi_j = \Phi_{j-1} \frac{d_{n-j}}{b_{n-j}} + \tan(\Phi_j - \Phi_{j-1})[\Phi_j - \Phi_{j-1}]^2 - \frac{(s + d)\cos \Phi_{j-1} \Phi_j}{h \cos(\Phi_j - \Phi_{j-1})^2}.
\]  
which is used in calculating \( B_n(\theta) \) defined in Eq. (25).

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APPENDIX: PROPERTIES OF THE \( \Psi \) FUNCTIONS

The \( \Psi_j(\theta) \) are defined on the interval \( 0 < \theta < \theta_c \) because \( \theta_n \) is restricted to this interval and ceases to be the foremost domino for larger values. The \( \Psi \) values are limited to \( 0 < \Psi_j(\theta) < \theta_c \). Figure 1 shows that the tilt angles all monotonically increase with increasing angle \( \theta \) of the foremost moving domino. So the functions \( \Psi_j(\theta) \) are monotonically increasing functions. They become flatter with increasing index \( j \) and converge (exponentially fast) to the value \( \theta_c \). All the properties of the functions \( \Psi_j(\theta) \) are implied by the recursion relation (1) together with \( \Psi_j(\theta) = \theta \) (by definition). The recursion relation for \( \Psi_j \) is obtained from Eq. (1) by the substitution \( \theta = \theta_{n-j} \).

\[
\sin[\Psi_j(\theta) - \Psi_{j-1}(\theta)] = \frac{s + d}{h} \cos \Psi_{j-1}(\theta) - \frac{d}{h}.
\]  

The functions are strongly interrelated. Not only can we calculate \( \Psi_j(\theta) \) from \( \Psi_{n-j} \) but \( \Psi_j(\theta) \) follows also from an arbitrary intermediate \( \Psi_k \) by \( \Psi_k \).

\[
\Psi_{n+1}(\theta) = \Psi_{k+1}(\theta), \quad \text{e.g.} \quad \Psi_j(\theta) = \Psi_{j-1}(\theta) + \Psi_{j+1}(\theta).
\]  

The relevance becomes evident when we express the right-hand side in terms of the moment arms defined in Eq. (19),

\[
\Psi_j = \Psi_{j-1}(a_{n-j}/b_{n-j}),
\]  
which implies that the frictionless forces conserve energy as shown in Sec. VI. Another differentiation of Eq. (A4) gives the recursion relation between the second derivatives \( \Phi_j \). It can be cast into the form

\[
\Phi_j = \Phi_{j-1} \frac{d_{n-j}}{b_{n-j}} + \tan(\Phi_j - \Phi_{j-1})[\Phi_j - \Phi_{j-1}]^2 - \frac{(s + d)\cos \Phi_{j-1} \Phi_j}{h \cos(\Phi_j - \Phi_{j-1})^2}.
\]  
which is used in calculating \( B_n(\theta) \) defined in Eq. (25).


The code for calculating the velocity of the domino effect can be found (with or without friction) at www.lorentz.leidenuniv.nl/lunchcalc/dominoes/.